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Double-logarithmic Scaling of the Structure Function F_2 at small x

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Abstract

Recent data on the structure function $F_2(x, Q^2)$ at small values of x are analysed and compared with theoretical expectations. It is shown that the observed rise at small x is consistent with a logarithmic increase, growing logarithmically also with Q^2. A stronger increase, which may be incompatible with unitarity when extrapolated to asymptotically small values of x, cannot be inferred from present data.

1 Introduction

The H1 and ZEUS collaborations at the *ep* collider HERA have published measurements of the structure function $F_2(x, Q^2)$ [1, 2, 3], which extend the range in x and Q^2 by two orders of magnitude as compared to previous fixed target experiments. A prominent rise of the structure function has been observed at small values of x, which has stimulated a variety of theoretical investigations.

The behaviour of the structure function $F_2(x, Q^2)$ for $x \to 0$, with Q^2 fixed, corresponds to the high-energy or Regge limit for virtual photon-proton scattering. The computation of the high-energy behaviour of cross sections in QCD is an important and still unsolved problem. Theoretical approaches to understand the Regge limit are almost exclusively based on perturbation theory. The standard evolution equations[1] predict a rise of the structure function F_2 at small x. This holds for the double-asymptotic solution of the evolution equations at large Q^2 and small x [5], as well as for the solution of the BFKL equation [6] which makes a prediction for the growth at small x for fixed Q^2. One expects that the rise of cross sections at large energies is eventually damped by screening corrections [7] leading to an asymptotic behaviour which, for proton-proton scattering, has to satisfy the Froissart bound [8].

The starting point of this paper is the examination of the small-x regime of the HERA data. For values of x below about 10^{-2} all measurements are compatible with a double-logarithmic behaviour in x and Q^2. We then address the question whether the observed rise of the structure function F_2 at small x is consistent with unitarity bounds. It is argued that a $\ln \frac{1}{x}$ increase, if persistent to asymptotically small x , may be compatible with constraints from unitarity. The double-logarithmic fit to the data is compared with the double-asymptotic form considered by Ball and Forte [9, 10].

2 Phenomenological analysis

The collaborations H1 and ZEUS have provided measurements at various values of Q^2 covering the range in x as shown in fig. 1. The measurements of both collaborations are compatible with each other. The 1994 data by the H1 collaboration [3] are used for the fits, since at present they are the most precise ones, and they also allow the distinction between correlated and uncorrelated contributions to the systematic uncertainties.

[1]For a review, see [4].

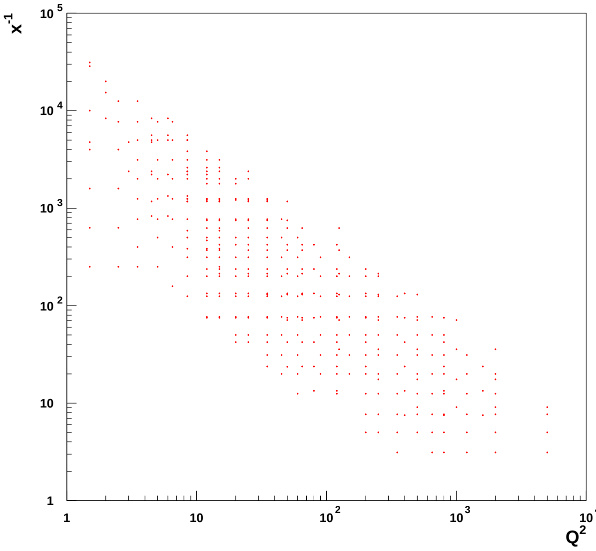

Figure 1: *(Q^2,$\frac{1}{x}$)-phasespace region covered by the H1 and ZEUS measurements of* F_2.

The behaviour of the structure function F_2 in the covered kinematical range is displayed in figs. 2 and 3 as function of x and Q^2, respectively. The Q^2-dependence for three bands in x, displayed in fig. 2, shows a logarithmic behaviour. The slopes are strongly x -dependent. For $x \simeq 0.05$ the slope vanishes and the structure function F_2 becomes independent of Q^2. Fig. 3 shows the x -behaviour of F_2 for three values of Q^2. The striking new feature of the HERA data is the prominent rise at values of x below about 10^{-2}. This small-x regime connects to the valence region, which was intensively investigated in previous low energy experiments.

In the following only the small-x data will be considered. The restricted phase space region is defined by

$$x < 0.010 \quad , \quad Q^2 > 5 \text{ GeV}^2 \quad . \tag{1}$$

The cut in x implies an effective cut of the high Q^2-data (cf. fig. 1).

It is a remarkable property of the data that in this small-x regime they are well described by a polynomial linear in $\log \frac{1}{x}$ for every measured Q^2-value,

$$F_2(x, Q^2) = u_0(Q^2) + u_1(Q^2) \log \frac{v(Q^2)}{x} \; . \tag{2}$$

Possible higher order terms in $\log \frac{1}{x}$ are statistically not significant and are consequently not considered. The quantities u_0 and u_1 are functions of Q^2, as well as the quantity $\log v$,

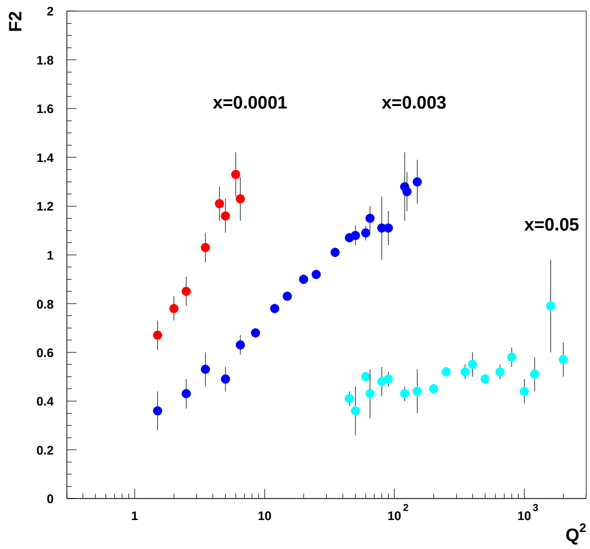

Figure 2: *$F(x\ ,Q^2)$ versus Q^2 for three values of x : upper points 0.0001, middle points 0.003, lower points 0.05.*

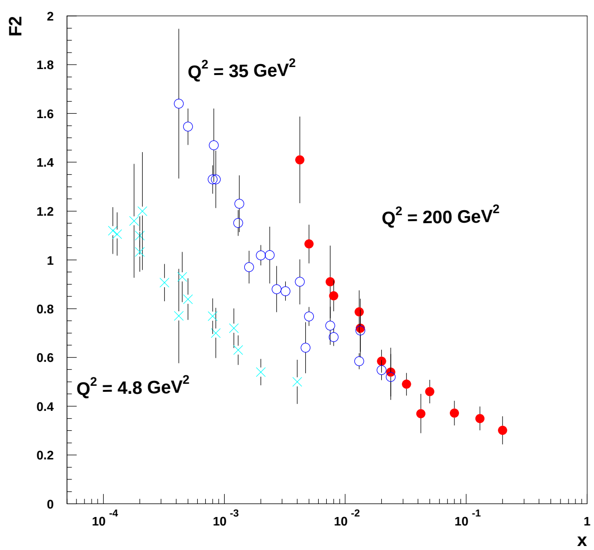

Figure 3: *$F(x\ ,Q^2)$ versus x for three values of Q^2 : 4.8, 35 and 200 GeV^2.*

which is uniquely defined as the weighted average $\langle \log \frac{1}{x} \rangle$. Using uncorrelated errors, also the uncertainties of u_0 and u_1 are uncorrelated. The numerical value of $v(Q^2)$ reflects the available x-range for any given Q^2 (see fig. 1), as well as the precision of the data. It is then possible to represent the whole body of data in the restricted phase space region, defined by (1), for each measured Q^2-value by three numbers,

$$v(Q^2) \ , \quad u_0(Q^2) \pm \delta u_0(Q^2) \ , \quad u_1(Q^2) \pm \delta u_1(Q^2) \ . \tag{3}$$

In terms of F_2 the two independent functions u_0 and u_1 represent average and slope,

$$\begin{aligned} u_0(Q^2) &= F_2(v(Q^2), Q^2) \ , \\ u_1(Q^2) &= \frac{\partial}{\partial \log \frac{1}{x}} F_2(x \ , Q^2) \ . \end{aligned} \tag{4}$$

The data on F_2 is consistent with a linear dependence in both $\log Q^2$ and $\log \frac{1}{x}$. Furthermore, its extrapolation to smaller values in $\log Q^2$ and $\log \frac{1}{x}$, respectively, suggests the existence of a common "fixpoint" (x_0,Q_0^2) (cf. figs. 3, 2). All this can be summarized in an ansatz for F_2 which is linear in the double-logarithmic scaling variable ξ,

$$\begin{aligned} F_2(x \ , Q^2) &= a + m \ \xi \ , \\ \xi &= \log \frac{Q^2}{Q_0^2} \log \frac{x_0}{x} \ . \end{aligned} \tag{5}$$

This simple form, when confronted with the data given in (3), implies

$$\begin{aligned} a + m \log \frac{Q^2}{Q_0^2} \log \frac{x_0}{v(Q^2)} &= u_0(Q^2) \ , \\ m \log \frac{Q^2}{Q_0^2} &= u_1(Q^2) \ . \end{aligned} \tag{6}$$

The first of these equations can be cast into the form

$$\frac{u_0 - a}{\log (x_0/v)} = u_1 \ , \tag{7}$$

thus allowing the comparison between the directly measured slope u_1 and the one obtained from the point (v, u_0) and the fixpoint (x_0,a). This is illustrated in fig. 4. The two parameter pairs (a,x_0) and (m,Q_0^2) are correlated, as is obvious from eq. (5). They are chosen as follows :

$$a = 0.078 \ , \ m = 0.364 \ , \ x_0 = 0.074 \ , \ Q_0^2 = 0.5 \ \mathrm{GeV}^2 \ . \tag{8}$$

Using the H1-data [3] with uncorrelated errors the measured structure function F_2 is plotted in fig. 5 as a function of the scaling variable ξ computed from x and Q^2 with the

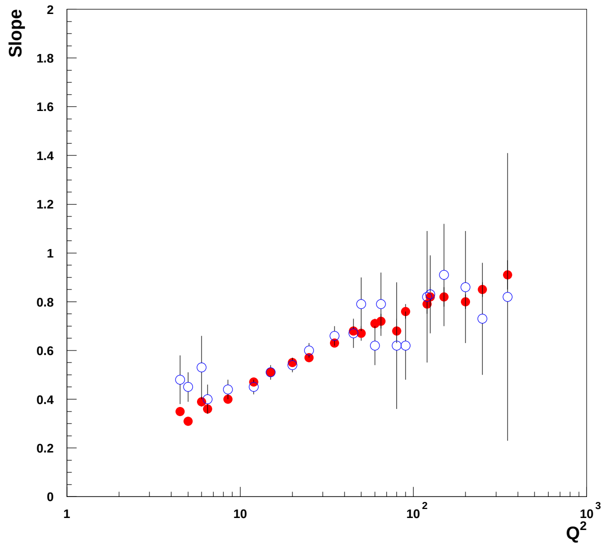

Figure 4: *The open circles represent the directly measured slopes* u_1*, the full circles are the induced slopes for* $x_0 = 0.074$ *and* $a = 0.078$*.*

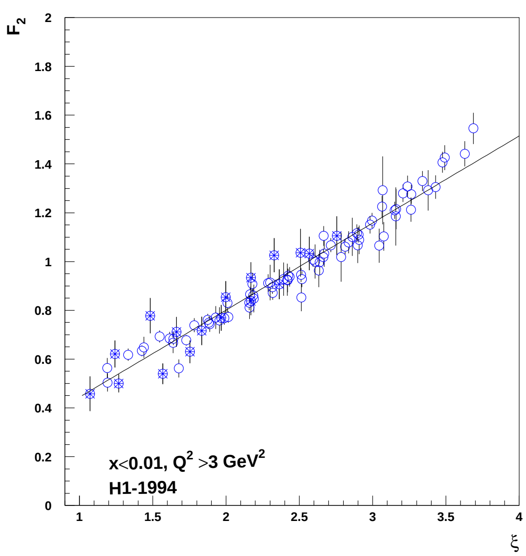

Figure 5: *The* F_2*-data of H1-1994 is plotted versus the scaling variable* ξ*. The open circles represent the data for* Q^2*above 5* GeV^2*; the crossed circles correspond to the data with* Q^2*between 3 and 5* GeV^2*.*

parameter values for x_0 and Q_0^2 as given in eq. (8). The χ^2/dof is 83/72. It turned out that even the data below 5 GeV^2 are well described by eq. (5). We have checked that also the other data on F_2 (c.f. [2, 1]) are well described by the same set of parameters.

3 Constraints from unitarity

No rigorous bound is known on the asymptotic behaviour of the structure function $F_2(x, Q^2)$ as $x \to 0$, with Q^2 fixed. However, within the framework of the parton model constraints on the allowed growth of parton densities at small x can be obtained by considering the Froissart bound [8] on the total proton-proton cross section at large energies,

$$\sigma_{tot}(s) \leq \frac{\pi}{m_\pi^2}\left(\ln\frac{s}{s_0}\right)^2 \ . \tag{9}$$

Here m_π is the pion mass, s is the center-of-mass energy squared and s_0 is an unknown constant. The total cross section measured at the Tevatron is of order $\pi/m_\pi^2 \sim 100$ mb.

The cross section for the production of particles with high tranverse momentum, $p_\perp^2 > \mu^2 \gg \Lambda_{QCD}^2$, can be calculated perturbatively. Clearly, this part of the cross section, σ_{pert}, must also satisfy the Froissart bound, i.e.

$$\sigma_{pert}(s, p_\perp^2 > \mu^2) < \sigma_{tot}(s) \ . \tag{10}$$

Here σ_{pert} is evaluated in terms of parton cross sections and parton densities. Eq. (10) then yields a consistency condition for the behaviour of the parton densities at small x.

In the parton model the dominant contribution to σ_{pert} is elastic gluon-gluon scattering. Consider the production of two gluons with transverse momentum $p_\perp \pm \Delta p_\perp$ and rapidities $y_3 \pm \Delta y$ and $y_4 \pm \Delta y$, respectively. Other partons in the final state are summed over. The corresponding cross section reads in lowest order perturbation theory (cf. [11])

$$\Delta\sigma_{gg} \simeq C\alpha_s(\mu)^2 x_1 g(x_1,\mu) x_2 g(x_2,\mu)\frac{1}{p_\perp^4}F(\theta^*)\Delta p_\perp^2 \Delta y^2 \ . \tag{11}$$

Here x_1 and x_2 are the momentum fractions of the gluons in the initial state, and $g(x,\mu)$ is the gluon density at scale μ; C is a constant and F is the averaged squared matrix element of the gluon-gluon cross section, which depends only on θ^*, the scattering angle in the gluon-gluon center-of-mass system. Note, that the multiplicity factor, which connects inclusive and exclusive cross sections, is $(1 + \mathcal{O}(\alpha_f))$ in eq. (11). In the special case $y_3 + y_4 = 0$, one has

$$x_1 = x_2 \equiv x = \frac{2p_T}{\sqrt{s}} \ . \tag{12}$$

The cross section (11) is a tiny fraction of the total cross section, $\Delta\sigma_{gg} < \alpha_s^2/\mu^2 \sim 1\mu$b. Its dependence on the center-of-mass energy s scales like $(xg(x,\mu))^2$, with $x = 2p_\perp/\sqrt{s}$. Clearly, if the gluon density satisfies the bound

$$xg(x,\mu) < B \ln\frac{1}{x} \ , \tag{13}$$

where B is a constant, then the corresponding bound on $\Delta\sigma_{gg}$ and the Froissart bound scale with s in the same way, i.e. like $(\ln s)^2$. Hence, $\Delta\sigma_{gg}$ will always remain far below the Froissart bound.

One may hope to derive a stronger bound on the gluon density based on a complete evaluation of the perturbative cross section σ_{pert}. Already the integration over the rapidities of the two final state gluons yields another factor of $(\ln s)^2$. Contributions with additional gluons in the final state yield further powers of $\ln s$ which eventually build up the full BFKL ladder [6]. It is conceivable that these perturbative corrections can be absorbed in a properly defined gluon density for which the bound (13) may then apply. A similar analysis could be carried out for deep-inelastic scattering where one has only one gluon in the initial state. In this way it may be possible to obtain a true bound on the structure function F_2 at small x. This, however, is beyond the scope of our paper. In any case, it is clear that with a logarithmic growth of the gluon density the perturbative cross section can be extrapolated several orders of magnitude beyond present center-of-mass energies before it possibly reaches the Froissart bound.

One can readily evaluate the contribution of photon-gluon fusion to the structure function F_2 for a gluon density saturating the bound (13). One finds (cf. [12]),

$$\Delta F_2(x,Q^2) = A + \frac{\alpha_s}{3\pi}\sum_q e_q^2 B \ln\frac{Q^2}{Q_0^2}\ln\frac{x_0}{x}, \tag{14}$$

where the sum extends over all quarks with masses small compared to Q, and the constant A depends on the renormalization scheme. Eq. (14) is identical to the fit (5) of F_2, obtained by the phenomenological analysis in sect. 2, if parameters are properly identified. ¿From the slope $m = 0.364$ one obtains $B \simeq 3$. At small x the gluon density is large, and the structure function is likely to be dominated by photon-gluon fusion.[2]

Eq. (14) relates the behaviour of F_2 at small x to a constraint on the gluon density obtained from the total proton-proton cross section. Our discussion has been based on a comparison of cross sections in perturbation theory to leading order, and the effect of higher order corrections is not clear. It is conceivable that, at fixed Q^2, $F_2(x,Q^2)$

[2]A similar fit to F_2, with $\ln 1/x$ replaced by a small power $x^{-\lambda}$ has been performed in [13].

continues to rise like $\ln\frac{1}{x}$ down to asymptotically small values of x. In fact, such a behaviour has been predicted by Bjorken. His starting point is an expression for the photon wave function renormalization constant derived by Gribov [14],

$$1 - Z_3 = \frac{\alpha}{3\pi}\int ds \frac{sR(s)}{(Q^2+s)^2}\ , \tag{15}$$

where $R(s)$ is the hadronic cross section in e^+e^--annihilation normalised to the μ-pair cross section. Based on the aligned-jet picture of deep inelastic scattering he then obtains for the inclusive structure function [15]

$$F_2(x,Q^2)\ \propto\ \bar{R}\ \ln\frac{1}{x}\ . \tag{16}$$

This corresponds to a gluon density saturating the bound (13). At present, however, we do not know whether eqs. (16) or (14) represent the correct behaviour of $F_2(x,Q^2)$ at asymptotically small values of x. There is neither a proof that such a behaviour will ever be reached nor can it be excluded that it might set in already at moderate values of x.

4 Comparison with double-asymptotic scaling

So far we have discussed the Regge limit, i.e. $x \to 0$ with Q^2 fixed. In this limit the double-logarithmic scaling form (5) of the structure function F_2 may be correct. However, in an appropriate simultaneous limit $Q^2 \to \infty$ and $x \to 0$ (see below), the double-logarithmic scaling form is expected to be incorrect. In this limit, for appropriate boundary conditions, the structure function is given by an asymptotic solution of the standard evolution equations [4], which was found more than 20 years ago [5], and which has recently been thoroughly studied by Ball and Forte [9, 10]. This solution is known to correspond to a summation of all terms of the form

$$\left(\alpha_s \ln\frac{Q^2}{\Lambda^2}\ln\frac{1}{x}\right)^n\ . \tag{17}$$

Hence, at small x, it increases faster than any power of $\ln\frac{1}{x}$. The double-logarithmic form (5) corresponds to the first term in this series. A comparison of these two expressions with data on F_2 therefore tests for the evidence of terms more singular than $\ln\frac{1}{x}$ at small x.

The asymptotic solution is conveniently expressed in terms of the variables [10]

$$\sigma = \left(\ln\frac{x_0}{x}\ln\left(\frac{\alpha_s(Q_0)}{\alpha_s(Q)}\right)\right)^{1/2}\quad ,\quad \rho = \left(\ln\frac{x_0}{x}\Big/\ln\left(\frac{\alpha_s(Q_0)}{\alpha_s(Q)}\right)\right)^{1/2}\ , \tag{18}$$

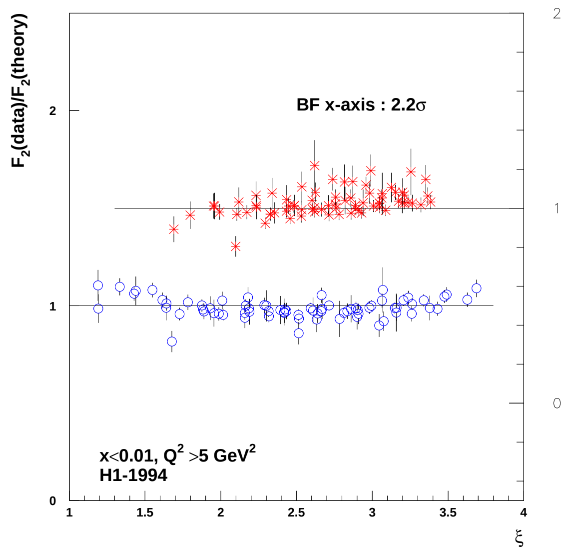

Figure 6: *Comparison of F_2-data with the Ball-Forte fit (right vertical scale) and the double-logarithmic fit (left vertical scale); the horizontal axis corresponds to 2.2 σ and ξ, respectively.*

where α_s is the two-loop QCD coupling,

$$\alpha_s(Q) = \frac{4\pi}{\beta_0 \ln \frac{Q^2}{\Lambda^2}} \left(1 - \frac{\beta_1}{\beta_0^2} \frac{\ln\ln \frac{Q^2}{\Lambda^2}}{\ln \frac{Q^2}{\Lambda^2}} \right) \ . \tag{19}$$

The first two coefficients of the β-function and other relevant parameters read [10]

$$\beta_0 = 11 - \frac{2}{3} n_f \ , \quad \beta_1 = 102 - \frac{38}{3} n_f \ , \quad \gamma^2 = \frac{12}{\beta_0} \ , \quad \delta = \frac{1}{\beta_0}\left(11 + \frac{2}{27} n_f\right) \ ,$$
$$\epsilon_+ = \frac{1}{\beta_0}\left(3\frac{\beta_1}{\beta_0} + \frac{103}{27} n_f\right) \ , \quad \epsilon_- = \frac{78}{\beta_0 \gamma^2} \ . \tag{20}$$

At large values of σ the structure function F_2 can now be written as [10]

$$F_2(x\ ,Q^2) \ = \ C\ \exp 2\gamma\sigma\ \exp\left(-\delta\frac{\sigma}{\rho} - \frac{1}{2}\ln\gamma\sigma - \ln\frac{\rho}{\gamma} \right)$$
$$\times \left(1 + (\epsilon_+ + \epsilon_-)\,\alpha_s(Q) - \epsilon_+\alpha_s(Q_0)\right) \frac{\rho}{\gamma}\left(1 + \mathcal{O}(\frac{\infty}{\sigma}) + \mathcal{O}(\frac{\infty}{\rho})\right) \ . \tag{21}$$

Here C is an unknown normalization constant. For large values of σ and ρ the first exponential gives the dominant contribution,

$$F_2(x, Q^2) \simeq C\ \exp\left(2\gamma\sigma\right) \ . \tag{22}$$

In this limit the structure function only depends on σ, which is referred to as double-asymptotic scaling.

Eq. (21) gives F_2 in terms of four a priori unknown constants, Λ, Q_0, x_0 and C. It is known that a proper choice of these parameters yields a good description of the measured data on F_2 in the range of small x and large Q^2 [10, 3]. Using the optimised values of [3], the constancy of the ratio between the data and the theoretical prediction (21) is clearly borne out in fig. 6 and quantified by $\chi^2/dof = 93/69$. Eq. (21) has been used for the number of flavours $n_f = 4$. A more sophisticated treatment incorporating Q^2-dependent threshold effects has not been attempted.

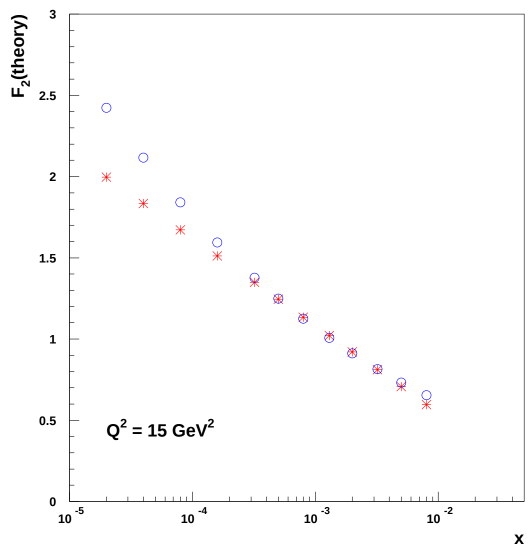

Figure 7: *The structure function F_2 for $Q^2 = 15\ GeV^2$ for the Ball-Forte fit (open circles) and the double-logarithmic fit (stars) extrapolated to values of x one decade below the HERA range.*

What does this mean concerning the validity of double-asymptotic scaling? As we have seen in sect. 2 (cf. also fig. 6), the data are equally well described by just the first term in the sum of double logarithms given in eq. (17). Hence, the characteristic feature of double-asymptotic scaling, a growth stronger than any power of $\ln \frac{1}{x}$ cannot be inferred from the present HERA data. This more singular behaviour should become visible if, at given Q^2, the range in x is extended by one order of magnitude. As fig. 7 illustrates, a clear difference is then expected between the double-logarithmic fit (5) and the double-asymptotic form (21).

5 Conclusions

We have shown that data on the structure function $F_2(x, Q^2)$ at small x and large Q^2 published up to date are consistent with a logarithmic growth in $\frac{1}{x}$ as well as Q^2. This is of interest for several reasons.

No rigorous bounds are known on the asymptotic behaviour of $F_2(x, Q^2)$ in the Regge limit, i.e. as $x \to 0$ with Q^2 fixed. However, according to our discussion in sect. 3, a logarithmic growth of F_2 can be extrapolated to smaller values of x by many orders of magnitude without getting into conflict with the Froissart bound. Such a logarithmic increase with $\frac{1}{x}$ may even be the correct asymptotic behaviour of the structure function.

In the double-asymptotic regime of small x and large Q^2 perturbative QCD predicts double-asymptotic scaling for the structure function F_2, given sufficiently soft input distributions. This implies a growth stronger than any power of $\ln \frac{1}{x}$. So far, however, the data are still consistent with double-logarithmic scaling corresponding to an increase like $\ln \frac{1}{x}$. This may mean that at HERA the small-x regime has not yet been reached, where the strong growth expected on the basis of perturbative QCD will become visible.

More precise measurements may differentiate between double-logarithmic scaling, double-asymptotic scaling and the even more singular BFKL power behaviour. As discussed in sect. 4, these differences should become manifest if, for given Q^2, the range in x is extended by one order of magnitude. This corresponds to an increase in the center-of-mass energy squared by one order of magnitude, which could be reached at future colliders, such as LEP$\otimes$LHC or at a 500 GeV Linear Collider$\otimes$HERA.

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